Harmonic distributions for equitable partitions of a hypercube

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Nov. 17. 2012

CONTENTS

Basic definitions on equitable partitions

- Q_n: the set of (binary) words of length n, called an n dimensional hypercube or an n-cube.
- $C \subseteq Q_n$: a binary code of length n
- For $u \in Q_n$, $\text{supp}(u) = \{i : u_i \neq 0\}$, $u = \text{supp}(u)$.
- $w(u) = |\text{supp}(u)| = |u|$, $d(u, v) = w(u - v)$.

Equitable partitions

• Let $\pi = \{C_1, C_2, \ldots, C_r\}$ be a partition of Q_n into $r\ (\geq 2)$ nonempty parts. We say that $\pi = \{C_1, C_2, \ldots, C_r\}$ is an equitable partition of Q_n with quotient matrix $B = (b_{ij})_{r\times r}$ if for all i and j, any word in C_i has exactly b_{ij} neighbors in C_j.

It should be mentioned that b_{ij} does not depend on the choice of words in C_i.

• D: the adjacent matrix of Q_n

1. $Spec(D) = \{n - 2|u| : u \in Q_n\}$.
2. $n \in Spec(B) \subseteq Spec(D)$.
3. B is diagonalizable.

Examples of an equitable bipartition

In the 3-cube, consider the following quotient matrices

- $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$
- $\begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$
- $\begin{pmatrix} 0 & n \\ 1 & n-1 \end{pmatrix}$, where $n = 2^m - 1, m \geq 2 \leftarrow$ a perfect code.

If $B = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$ is a quotient matrix of Q_n, then $Spec(B) = \{n, i-k\}$.

Completely regular codes

For $C \subseteq Q_n$ and $u \in Q_n$,
- $d(u, C) = \min\{d(u, v) : v \in C\}$.
- $\rho(C) = \rho = \max\{d(u, C) : u \in Q_n\}$: the covering radius of C.
- $C(l) = \{u \in Q_n : d(u, C) = l\}$. In particular, $C(0) = C$.

- A partition $\pi(C) = \{C(0), C(1), \ldots, C(\rho)\}$ of Q_n : the distance partition of C.

- $C \subseteq Q_n$: a completely regular code with covering radius ρ if the distance partition $\pi(C)$ is equitable.

perfect codes $\hookrightarrow$ completely regular codes $\hookrightarrow$ equitable partitions.

Harmonic spaces

- S_k: the sphere of radius k centered at zero word.

- $\mathbb{R}S_k$: the free real vector space spanned by S_k.
- For $f \in \mathbb{R}S_k$,

$$f = \sum_{u \in S_k} f(u)u.$$

- For $f \in \mathbb{R}S_k$ and $S \subseteq \{1, 2, \ldots, n\}$, we define

$$\tilde{f}(u) = \sum_{u \in S} \left(\sum_{v \in S_k, v \subseteq u} f(v) \right) u = \sum_{v \in S_k, v \subseteq u} f(v).$$

Harmonic spaces

• The differentiation γ is the operator defined by linearity from

$$\gamma : S_k \longrightarrow S_{k-1}, u \mapsto \sum_{v \in S_{k-1}, v \subseteq u} v.$$

• $Harm_k$ is defined by the kernel of γ, i.e., $Harm_k = Ker(\gamma|_{\mathbb{R}S_k}) \ni$ a harmonic function of degree k.

• $h \in Harm_k$ iff $\sum_{u \in S_k, v \subseteq u} h(u) = 0$ for all $v \in S_k$.

Harmonic distributions

$f, g : Q_n \to \mathbb{R}$.

• The harmonic weight enumerator for f of Q_n (attached to a harmonic function h of degree k):

$$W_{f,h}(x,y) = \sum_{u \in Q_n} f(u)\tilde{h}(u)x^{n-|u|-k}y^{|u|-k}.$$

If f is the characteristic function of a code C, then we simply denote it by $W_{C,h}(x,y)$, first introduced by Bachoc.

• Harmonic distributions:
(i) The harmonic distribution $(A_0(f;h), A_1(f;h), \ldots, A_n(f;h))$ for f of Q_n

$$A_i(f;h) = \sum_{u \in S_i} f(u)\tilde{h}(u) = \sum_{u \in S_i} f(u)h(u).$$

(ii) The harmonic distribution $(A_0(f,g;h), A_1(f,g;h), \ldots, A_n(f,g;h))$ for f and g of Q_n

$$A_i(f,g;h) = \sum_{u,v \in S_i} f(u)g(v)\tilde{h}(u+v).$$

Harmonic distributions for eigenfunction of D

Proposition

Let f be an eigenfunction of D with eigenvalue λ, C a code in Q_n and h in $Harm_k$. Then

$$W_{f,h}(x,y) = (x+y)^{\frac{n+\lambda}{2}-k}(x-y)^{\frac{n-\lambda}{2}-k}\sum_{u\in S_k} f(u)h(u),$$

$$\sum_{u\in Q_n, v\in C} f(u)\tilde{h}(u+v)x^{n-|u+v|-k}y^{|u+v|-k}$$

$$= (x+y)^{\frac{n+\lambda}{2}-k}(x-y)^{\frac{n-\lambda}{2}-k}\sum_{u\in Q_n, v\in C, u+v\in S_k} f(u)\tilde{h}(u+v).$$

Orthogonal array, t-design

• An $M \times n$ matrix A with entries from Q_1 is called an orthogonal array of size M, n constraints, 2 levels, strength t and index λ if any set of columns of A contains all 2^k possible vectors exactly λ times.

• A collection $\mathcal{B}$ of k-subsets of $\{1, 2, \ldots, n\}$ is a t-design if every t-element subset lies in a constant number of elements in $\mathcal{B}$.

• Bachoc made use of the harmonic weight enumerator for a linear code to connect with t-designs based on the result of Delsarte.

Harmonic distributions for equitable partitions

Theorem

Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition of Q_n with quotient matrix B. For $j = 1, 2, \ldots, r$, let $(\alpha_{j1}, \alpha_{j2}, \ldots, \alpha_{jr})^T$ be an eigenvector of B with eigenvalue λ_j. Let $h \in Harm_k$. Then for $s, t = 1, 2, \ldots, r$, we have

$$W_{C_s,h}(x,y) = \sum_{j=1}^{r}\sum_{i=1}^{r} \beta_{si}\alpha_{ij}(x+y)^{\frac{1}{2}(n+\lambda_i)-k}(x-y)^{\frac{1}{2}(n-\lambda_i)-k} \sum_{u \in C_j \cap S_k} h(u),$$

$$\sum_{u \in C_s, v \in C_t} \tilde{h}(u+v)x^{n-|u+v|-k}y^{|u+v|-k}$$
$$= \sum_{j=1}^{r}\sum_{i=1}^{r} \beta_{si}\alpha_{ij}(x+y)^{\frac{n+\lambda_i}{2}-k}(x-y)^{\frac{n-\lambda_i}{2}-k} \sum_{u \in C_j, v \in C_t, u+v \in S_k} \tilde{h}(u+v),$$

where β_{ij} is an (i,j) entry of $[\alpha_{ij}]^{-1}$.

Orthogonal array

Theorem

Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition of Q_n. Then every cell forms an orthogonal array of strength at least

$$\frac{n-\lambda}{2} - 1,$$

where λ is the second largest eigenvalue of B.

• We remark that we can precisely compute the strength of cells defending on the membership of the zero word.

t-design

Theorem

Let $(C_1, C_2, \ldots, C_r)$ be an equitable partition of Q_n. Then the set of words of any fixed weight in each cell C_i forms a t-design if and only if $\sum_{C_i \cap S_k} h(u)$ is vanishing for $i = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, t$. In particular, if the size of C_i in S_k is either vanishing or $\binom{n}{k}$ for $i = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, t$, then the set of words of any fixed weight in every cell C_i forms a t-design.

Example

(1) Let (C_1, C_2) be an equitable bipartition of Q_n with quotient matrix $B = \begin{bmatrix} s & n-s \\ t & n-t \end{bmatrix}$. The eigenvalues of B are $n, s-t$ and

$$[\alpha_{ij}] = \begin{bmatrix} 1 & 1 \\ -n+s & t \end{bmatrix}, \ [\beta_{ij}] = \frac{1}{n-s+t} \begin{bmatrix} t & -1 \\ n-s & 1 \end{bmatrix}.$$

- Every cell forms an orthogonal array of strength $\frac{n-s+t}{2} - 1$.

If $k \geq 1$, then

$$W_{C_1,h}(x,y) = \frac{t}{n-s+t}(x+y)^{n-k}(x-y)^{-k} \sum_{u \in C_1 \cap S_k} h(u) + \frac{n-s}{n-s+t}(x+y)^{\frac{n+s-t}{2}-k}(x-y)^{\frac{n-s+t}{2}-k} \sum_{u \in C_1 \cap S_k} h(u),$$

$$W_{C_2,h}(x,y) = \frac{t}{n-s+t}(x+y)^{n-k}(x-y)^{-k} \sum_{u \in C_1 \cap S_k} h(u) - \frac{n-s}{n-s+t}(x+y)^{\frac{n+s-t}{2}-k}(x-y)^{\frac{n-s+t}{2}-k} \sum_{u \in C_1 \cap S_k} h(u).$$

• If $d(C_1) = 2$, then $W_{C_i,h}(x,y) = 0$ for $i = 1,2, k = 1$. In this case, the set of words of any fixed weight in the cells $C_i, i = 1,2$ forms a 1-design.

• If $d(C_1) = 3$, then the set of words of any fixed weight in the cells $C_i, i = 1,2, k = 1,2$ forms a 2-design.